On spectrum of a Schrödinger operator with a fast oscillating compactly supported potential

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Abstract

We study the phenomenon of an eigenvalue emerging from essential spectrum of a Schrödinger operator perturbed by a fast oscillating compactly supported potential. We prove the sufficient conditions for the existence and absence of such eigenvalue. If exists, we obtain the leading term of its asymptotics expansion.

The present work is devoted to the study of the spectrum of the operator

$$H_\varepsilon := -\frac{d^2}{dx^2} + V\left(x, \frac{x}{\varepsilon}\right)$$

in $L_2(\mathbb{R})$ with domain $W_2^2(\mathbb{R})$. Here ε is a small positive parameter, $V(x,\xi)$ is a complex-valued 1-periodic on ξ function belonging to $C^\infty(\mathbb{R}^2)$, such that for all $\xi\in\mathbb{R}$ the support $\operatorname{supp} V(\cdot,\xi)$ is bounded uniformly on ξ.

The operator $H_0 := -\dfrac{d^2}{dx^2}$ in $L_2(\mathbb{R})$ with domain $W_2^2(\mathbb{R})$ is self-adjoint, its discrete spectrum is empty while the essential one coincides with the semi-axis $[0,+\infty)$. The multiplication operator by the function $V\left(x,\frac{x}{\varepsilon}\right)$ is H_0-compact. This is why by Theorems 1.1, 5.35 in Chapter IV of [1] the operator H_ε is closed for all ε and the essential spectra of the operator H_ε and H_0 are same.

The aim of the work is to study the existence and the asymptotics behaviour of the eigenvalues of the operator H_ε, tending to zero as $\varepsilon\to 0$, in the case the mean value of the function $V(x,\cdot)$ over period is zero for all $x\in\mathbb{R}$. Such eigenvalues can be also regarded as emerging from the border of the essential spectrum when perturbing the operator H_0 by the potential $V\left(x,\frac{x}{\varepsilon}\right)$. We note that the phenomenon of the eigenvalues emerging from the border of a essential spectrum under the perturbation by the potential was treated in [2]–[5] for the potentials of the form $\varepsilon U(x)$, where $U(x)$ is a sufficiently rapidly decaying real potential. This phenomenon was also considered in [6] for the perturbation $\varepsilon L_\varepsilon$, where $L_\varepsilon : W_{2,loc}^2(\mathbb{R}) \to L_2(\mathbb{R};Q)$ is an arbitrary operator obeying a uniform on ε inequality $\|L_\varepsilon u\|_{L_2(\mathbb{R})} \leqslant C\|u\|_{W_2^2(Q)}$ for a finite interval Q in the axis, and

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$L_2(\mathbb{R};Q)$ is a subset of the functions from $L_2(\mathbb{R})$ whose supports are in $\overline{Q}$. In [2]–[6] for the mentioned perturbations the existence of the eigenvalues emerging from the border of the essential spectrum was studied. If exists, the leading term of the asymptotics expansions for an eigenvalue was constructed. Clearly, the perturbation considered in the articles cited do not include the potential $V\left(x, \frac{x}{\varepsilon}\right)$. Moreover, the perturbation described by this potential is not regular in the sense that the multiplication operator by this potential does not tend to zero uniformly as $\varepsilon \to 0$.

Let a segment $M = [x_0, x_1]$ be such that for all $\xi \in \mathbb{R}$ the inclusion $\operatorname{supp} V(\cdot, \xi) \subseteq M$ holds true. By W we denote the set of 1-periodic on ξ functions $u(x, \xi)$ from $C^\infty(\mathbb{R}^2)$ such that $\operatorname{supp} u(\cdot, \xi) \subseteq M$ for all $\xi \in \mathbb{R}$. For a function from W its mean value over period will be indicated as

$$\langle u(x, \cdot)\rangle := \int\limits_0^1 u(x, \xi)\, \mathrm{d}\xi.$$

Lemma. *For each function $u \in W$ and each number $n \geqslant 1$ the equality*

$$\int\limits_{\mathbb{R}} u\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int\limits_{\mathbb{R}} \langle u(x, \cdot)\rangle \, \mathrm{d}x + \mathcal{O}(\varepsilon^n)$$

holds true.

Proof. It is sufficient to prove the statement of the lemma in the case $\langle u(x, \cdot)\rangle \equiv 0$, since the general case is reduced to this one by the change $\widetilde{u}(x, \xi) = u(x, \xi) - \langle u(x, \cdot)\rangle$. Let $\langle u(x, \cdot)\rangle \equiv 0$. We denote

$$P[u](x, \xi) := \int\limits_0^\xi u(x, \tau)\, \mathrm{d}\tau + \int\limits_0^1 \tau u(x, \tau)\, \mathrm{d}\tau.$$

Then $P[u] \in W$ and $\langle P[u](x, \cdot)\rangle = 0$. Bearing in mind an obvious equality

$$u\left(x, \frac{x}{\varepsilon}\right) = \varepsilon \frac{d}{dx} P[u]\left(x, \frac{x}{\varepsilon}\right) - \varepsilon \frac{\partial P[u]}{\partial x}\left(x, \frac{x}{\varepsilon}\right),$$

we get

$$\int\limits_{\mathbb{R}} u\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = -\varepsilon \int\limits_{\mathbb{R}} \frac{\partial P[u]}{\partial x}\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x. \tag{1}$$

Since $\dfrac{\partial P[u]}{\partial x} \in W$ $\left\langle \dfrac{\partial P[u]}{\partial x}(x, \cdot)\right\rangle \equiv 0$, it follows that the equality (1) is applicable to the function $\dfrac{\partial P[u]}{\partial x}$ as well. Applying this equality as many times as needed, we arrive at the statement of the lemma. $\square$

We denote

$$k_2 := \frac{1}{2}\int\limits_{\mathbb{R}} \left\langle \big(P[V](x,\cdot)\big)^2 \right\rangle \mathrm{d}x. \tag{2}$$

The main result of the work is the following

Theorem. *Let*

$$\langle V(x,\cdot)\rangle \equiv 0. \tag{3}$$

Then

1. *If* $\operatorname{Re} k_2 > 0$*, then there exists the unique eigenvalue* λ_ε *of the operator* H_ε*, tending to zero as* $\varepsilon \to 0$*. This eigenvalue is simple and its asymptotics is of the form:*
$$\lambda_\varepsilon = -\varepsilon^4 k_2^2 + \mathcal{O}(\varepsilon^5). \tag{4}$$
2. *If* $\operatorname{Re} k_2 < 0$*, then the operator* H_ε *has no eigenvalues tending to zero as* $\varepsilon \to 0$*.*

. It is easy to check that the function

$$v(x,\xi) := \int\limits_0^{\xi} (\xi-\tau)V(x,\tau)\,\mathrm{d}\tau + \xi\int\limits_0^{1} \tau V(x,\tau)\,\mathrm{d}\tau + v_1(x)$$

is a solution to the equation

$$\frac{\partial^2 v}{\partial \xi^2} = V, \tag{5}$$

moreover,

$$\frac{\partial v}{\partial \xi} = P[V]. \tag{6}$$

By (3) and the belonging $V \in W$ the function v is 1-periodic on ξ, and, therefore, is bounded uniformly on $(x,\xi) \in \mathbb{R}^2$. We assume that $v_1 \in C^\infty(\mathbb{R})$ and $\operatorname{supp} v_1 \subseteq M$; then $v \in W$.

Let $q_\varepsilon(x,\xi) := 1 + \varepsilon^2 v(x,\xi)$. The multiplication operator by the function $\widetilde{q}_\varepsilon(x) := q_\varepsilon\left(x, \frac{x}{\varepsilon}\right)$ (we denote is by Q_ε) maps $L_2(\mathbb{R})$ in a one-to-one way onto itself. This is why the eigenvalues of the operator H_ε coincide with those of the operator $Q_\varepsilon^{-1} H_\varepsilon Q_\varepsilon$. By (5) and the definition of $\widetilde{q}_\varepsilon(x)$ the representation $Q_\varepsilon^{-1} H_\varepsilon Q_\varepsilon = H_0 - \varepsilon L_\varepsilon$ is valid, where

$$L_\varepsilon = \varepsilon \frac{2}{\widetilde{q}_\varepsilon(x)} \frac{d}{dx} v\left(x, \frac{x}{\varepsilon}\right) \frac{d}{dx} - \frac{f_\varepsilon\left(x,\frac{x}{\varepsilon}\right)}{\widetilde{q}_\varepsilon(x)}, \quad f_\varepsilon = \varepsilon V v - \varepsilon \frac{\partial^2 v}{\partial x^2} - 2\frac{\partial^2 v}{\partial x \partial \xi}. \tag{7}$$

Since V, $v \in W$, it follows that the supports of the coefficients of the operator L_ε lie in M for all values of ε, and the operator $L_\varepsilon : W_{2,loc}^2(\mathbb{R}) \to L_2(\mathbb{R}; M)$ meets a uniform on ε inequality

$$\|L_\varepsilon u\|_{L_2(\mathbb{R})} \leqslant C\|u\|_{W_2^2(M)}. \tag{8}$$

We denote

$$m_\varepsilon^{(1)} := \int\limits_{\mathbb{R}} L_\varepsilon[1]\,\mathrm{d}x, \quad m_\varepsilon^{(2)} := \int\limits_{\mathbb{R}} L_\varepsilon\left[\int\limits_{\mathbb{R}} |x-t| L_\varepsilon[1]\,\mathrm{d}t\right]\mathrm{d}x, \quad k_\varepsilon := \frac{\varepsilon}{2}m_\varepsilon^{(1)} + \frac{\varepsilon^2}{2}m_\varepsilon^{(2)}. \tag{9}$$

The estimate (8) begin valid for the operator L_ε, Theorem 1 of the work [6] implies that if

$$k_\varepsilon = \varepsilon c_1 + \varepsilon^2 c_2 + \mathcal{O}(\varepsilon^3), \quad c_1, c_2 = \mathrm{const}, \tag{10}$$

then the sufficient condition the operator $(H_0 - \varepsilon L_\varepsilon)$ to have the eigenvalue tending to zero as $\varepsilon \to 0$ is the inequality $\mathrm{Re}(c_1 + \varepsilon c_2) > 0$, while the sufficient condition of the absence is the inequality $\mathrm{Re}(c_1 + \varepsilon c_2) < 0$. If $\mathrm{Re}(c_1 + \varepsilon c_2) > 0$, then the operator $(H_0 - \varepsilon L_\varepsilon)$ has the unique eigenvalue tending to zero, this eigenvalue is simple and the equality $\lambda_\varepsilon = -(\varepsilon c_1 + \varepsilon^2 c_2)^2 + \mathcal{O}(c_1\varepsilon^4 + \varepsilon^5)$ holds true.

Thus, in order to prove the theorem it is sufficient to establish the equality (10) $c_1 = 0$, $c_2 = k_2$. Let us prove it. For the sake of simplicity of calculations we set

$$v_1(x) := \frac{1}{2}\int\limits_0^1 (\tau - \tau^2) V(x,\tau)\,\mathrm{d}\tau.$$

It is easy to see that in this case

$$\langle v(x,\cdot)\rangle = 0, \tag{11}$$

and, therefore,

$$\left\langle \frac{\partial^2 v}{\partial x^2}(x,\cdot)\right\rangle = \left\langle \frac{\partial^2 v}{\partial x \partial \xi}(x,\cdot)\right\rangle = 0. \tag{12}$$

We denote $\widetilde{f}_\varepsilon(x) := f_\varepsilon\left(x, \frac{x}{\varepsilon}\right)$. From (9), (7) and the definition of $\widetilde{q}_\varepsilon$ by Lemma and the equalities (12), (5), (6) it follows that

$$\begin{aligned}
\frac{\varepsilon m_\varepsilon^{(1)}}{2} &= -\frac{\varepsilon}{2}\int\limits_{\mathbb{R}} \frac{\widetilde{f}_\varepsilon(x)}{\widetilde{q}_\varepsilon(x)}\,\mathrm{d}x = -\frac{\varepsilon^2}{2}\int\limits_{\mathbb{R}} V\left(x, \frac{x}{\varepsilon}\right) v\left(x, \frac{x}{\varepsilon}\right)\mathrm{d}x + \mathcal{O}(\varepsilon^3) \\
&= -\frac{\varepsilon^2}{2}\int\limits_{\mathbb{R}} \langle V(x,\cdot) v(x,\cdot)\rangle\,\mathrm{d}x + \mathcal{O}(\varepsilon^3) = -\frac{\varepsilon^2}{2}\int\limits_{\mathbb{R}} \left\langle v(x,\cdot)\frac{\partial^2 v}{\partial \xi^2}(x,\cdot)\right\rangle \mathrm{d}x + \mathcal{O}(\varepsilon^3) \\
&= \frac{\varepsilon^2}{2}\int\limits_{\mathbb{R}} \left\langle \left(\frac{\partial v}{\partial \xi}(x,\cdot)\right)^2\right\rangle \mathrm{d}x + \mathcal{O}(\varepsilon^3) = \varepsilon^2 k_2 + \mathcal{O}(\varepsilon^3),
\end{aligned} \tag{13}$$

where k_2 is from (2).

Since by the definition of $\widetilde{q}_\varepsilon$

$$\frac{d}{dx}\ln \widetilde{q}_\varepsilon(x) = \frac{\varepsilon^2}{\widetilde{q}_\varepsilon(x)}\frac{d}{dx} v\left(x, \frac{x}{\varepsilon}\right),$$

integrating by parts and taking into account that for each function $g \in C_0(\mathbb{R})$ the equality

$$\frac{d^2}{dx^2}\int\limits_{\mathbb{R}} |x-t|g(t)\,\mathrm{d}t = 2g(x)$$

is valid, from (9), (7) we obtain:

$$\begin{aligned}\frac{\varepsilon^2 m_\varepsilon^{(2)}}{2} &= -\varepsilon\int\limits_{\mathbb{R}} \ln\widetilde{q}_\varepsilon(x)\frac{d^2}{dx^2}\int\limits_{\mathbb{R}} |x-t|\frac{\widetilde{f}_\varepsilon(t)}{\widetilde{q}_\varepsilon(t)}\,\mathrm{d}t\,\mathrm{d}x + \frac{\varepsilon^2}{2}\int\limits_{M^2}\frac{|x-t|\widetilde{f}_\varepsilon(x)\widetilde{f}_\varepsilon(t)}{\widetilde{q}_\varepsilon(x)\widetilde{q}_\varepsilon(t)}\,\mathrm{d}t\,\mathrm{d}x\\ &= -2\varepsilon\int\limits_{\mathbb{R}}\frac{\widetilde{f}_\varepsilon(x)\ln\widetilde{q}_\varepsilon(x)}{\widetilde{q}_\varepsilon(x)}\,\mathrm{d}x + \varepsilon^2\int\limits_{x_0}^{x_1}\frac{\widetilde{f}_\varepsilon(x)}{\widetilde{q}_\varepsilon(x)}\left(\int\limits_{x_0}^{x}\frac{(x-t)\widetilde{f}_\varepsilon(t)}{\widetilde{q}_\varepsilon(t)}\,\mathrm{d}t\right)\,\mathrm{d}x.\end{aligned}$$

The first integral in the right hand side of the equality obtained is of order $\mathcal{O}(\varepsilon^3)$. Integrating by parts in the second integral we deduce:

$$\begin{aligned}&\int\limits_{x_0}^{x_1}\frac{\widetilde{f}_\varepsilon(x)}{\widetilde{q}_\varepsilon(x)}\left(\int\limits_{x_0}^{x}\frac{(x-t)\widetilde{f}_\varepsilon(t)}{\widetilde{q}_\varepsilon(t)}\,\mathrm{d}t\right)\,\mathrm{d}x = \int\limits_{x_0}^{x_1} x\widetilde{f}_\varepsilon(x)\left(\int\limits_{x_0}^{x}\widetilde{f}_\varepsilon(t)\,\mathrm{d}t\right)\,\mathrm{d}x-\\ &-\int\limits_{x_0}^{x_1}\widetilde{f}_\varepsilon(x)\left(\int\limits_{x_0}^{x} t\widetilde{f}_\varepsilon(t)\,\mathrm{d}t\right)\,\mathrm{d}x + \mathcal{O}(\varepsilon^2) = 2\int\limits_{x_0}^{x_1} x\widetilde{f}_\varepsilon(x)\left(\int\limits_{x_0}^{x}\widetilde{f}_\varepsilon(t)\,\mathrm{d}t\right)\,\mathrm{d}x-\\ &-\int\limits_{\mathbb{R}} x\widetilde{f}_\varepsilon(x)\,\mathrm{d}x\int\limits_{\mathbb{R}}\widetilde{f}_\varepsilon(x)\,\mathrm{d}x + \mathcal{O}(\varepsilon^2) = -\int\limits_{x_0}^{x_1}\left(\int\limits_{x_0}^{x}\widetilde{f}_\varepsilon(t)\,\mathrm{d}t\right)^2\,\mathrm{d}x+\\ &+\int\limits_{\mathbb{R}}(x_1-x)\widetilde{f}_\varepsilon(x)\,\mathrm{d}x\int\limits_{\mathbb{R}}\widetilde{f}_\varepsilon(x)\,\mathrm{d}x + \mathcal{O}(\varepsilon^2).\end{aligned}$$

It follows from the definition of the function $\widetilde{f}_\varepsilon$, the formula for f_ε from (7), the equalities (12) and Lemma that the second term in the right hand side of the last equality is of order $\mathcal{O}(\varepsilon^2)$. Taking into account the definition of the function f_ε once again as well as the equality

$$\frac{\partial^2 v}{\partial x\partial\xi}\left(x,\frac{x}{\varepsilon}\right) = \varepsilon\frac{d}{dx}\frac{\partial v}{\partial x}\left(x,\frac{x}{\varepsilon}\right) - \varepsilon\frac{\partial^2 v}{\partial x^2}\left(x,\frac{x}{\varepsilon}\right),$$

we obtain

$$-\int\limits_{x_0}^{x_1}\left(\int\limits_{x_0}^{x}\widetilde{f}_\varepsilon(t)\,\mathrm{d}t\right)^2\,\mathrm{d}x = -4\int\limits_{M}\left(\int\limits_{x_0}^{x}\frac{\partial^2 v}{\partial x\partial\xi}\left(t,\frac{t}{\varepsilon}\right)\,\mathrm{d}t\right)^2\,\mathrm{d}x + \mathcal{O}(\varepsilon^2) = \mathcal{O}(\varepsilon^2).$$

Thus,

$$\int\limits_{x_0}^{x_1} \frac{\widetilde{f}_\varepsilon(x)}{\widetilde{q}_\varepsilon(x)} \left(\int\limits_{x_0}^{x} \frac{(x-t)\widetilde{f}_\varepsilon(t)}{\widetilde{q}_\varepsilon(t)} \,\mathrm{d}t \right) \mathrm{d}x = \mathcal{O}(\varepsilon^2),$$

hence, $\varepsilon^2 m_\varepsilon^{(2)} = \mathcal{O}(\varepsilon^4)$, and the equality (10) $c_1 = 0$, $c_2 = k_2$ now follows from (9) and (13). $\square$

It also follows from Theorem 1 of the paper [6] that all the eigenvalues of the operator H_ε except the one tending to zero (if they exist) must tend to infinity as $\varepsilon \to 0$. If V is a non-zero real-valued function, then all the eigenvalues of the operator H_ε are real and negative. Moreover, in this case k_2 is real and positive, hence, the operator H_ε has the unique eigenvalue, this eigenvalue is simple, tends to zero and has the asymptotics (4).

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